

# A NEW WAY TO EVALUATE MOY GRAPHS

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**ABSTRACT.** This paper is concerned with evaluation of  $\mathfrak{sl}_N$ -webs from a graph-theoretical point of view: We give an interpretation of the evaluation of  $\mathfrak{sl}_N$  webs in terms of colorings. This is very close from the approach of Cautis, Kamnitzer and Morrison, but we provide a non-local and algebra-free definition of the degree associated with a coloring. In particular we do not use skew Howe duality. As a counter-part we are only concerned with closed webs. We prove that this new evaluation coincides with the classical evaluation of MOY graphs by checking some skein relations. As a consequence, we prove a formula which relates the  $\mathfrak{sl}_N$  and  $\mathfrak{sl}_{N-1}$ -evaluations of MOY graphs.

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## 1. INTRODUCTION

MOY graphs and MOY graph evaluation have been introduced by Murakami Ohtsuki and Yamada [MOY98] to provide a combinatorial and computational approach to the  $\mathfrak{sl}_N$ -invariant of links [RT90]. The edges of these graphs are meant to represent some wedge powers of the fundamental representation of the Hopf algebra  $U_q(\mathfrak{sl}_N)$  and the vertices correspond to some intertwiners. A MOY graph can therefore be interpreted as an endomorphism of  $\mathbb{C}(q)$  the trivial  $U_q(\mathfrak{sl}_N)$ -module. The evaluation of a MOY graph is the image of 1 under this endomorphism.

Murakami, Ohtsuki and Yamada gave a combinatorial way to evaluate MOY graphs using colorings and state sums and gave some skein relations satisfied by the evaluation. Kim [Kim03] and Morrison [Mor07] conjectured that (a Karoubi-completion of) the category of MOY graphs is equivalent to that of finite dimensional  $U_q(\mathfrak{sl}_N)$ -modules. This has been proved by Cautis, Kamnitzer and Morrison [CKM14].

The combinatorial evaluation of MOY graphs has been used to study the categorification of the  $\mathfrak{sl}_N$ -invariant (see for example [KR08a, KR08b, Wu14, MSV09, LZ14, QR14, CKM14]).

In this paper, we give an alternative definition of the evaluation of MOY graphs. This new definition is very close from that of [CKM14]: just like Cautis, Kamnitzer and Morrison, we count all possible colorings of a MOY graphs taking into account a certain degree. As we only work with closed MOY graphs, many simplifications

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occur, giving us a global definition of the degree (this implies in particular that we do not need their *tags*). In other word, this paper can be seen as a combinatorial rewriting of part of [CKM14] in the case of closed MOY graphs.

From this new evaluation, one can deduce a skein relation which relates  $\mathfrak{sl}_N$  and  $\mathfrak{sl}_{N-1}$ -evaluations of MOY graphs (this formula can be as well derived from the diagrammatic description of the Gel'fand-Tsetlin functor in the PhD thesis of Morrison [Mor07, Chapter 4]).

We hope that our fully-combinatorial approach to the evaluation of closed MOY graphs can be applied to foam-theoretic categorifications of the  $\mathfrak{sl}_N$ -invariant. In particular, we think that it could helpful to get rid of the “ladder” formalism used, see for example [QR14].

**Organization of the paper.** In section 2, we define our evaluation  $\langle \cdot \rangle_{\text{col}}$  of  $\mathfrak{sl}_N$ -webs and we state in Theorem 2.6 that our evaluation agrees with the one defined in [MOY98]. We explain that in order to prove the theorem, it is enough to show that  $\langle \cdot \rangle_{\text{col}}$  satisfies some skein relations.

In section 3, we introduce degrees of partitions of ordered sets and show a relatively technical lemma about this notion which is the key point of the proof of Theorem 2.6.

In section 4, we prove that  $\langle \cdot \rangle_{\text{col}}$  satisfies the skein relations. Finally in section 5, we state and prove Proposition 5.3 which related the  $\mathfrak{sl}_N$  evaluation and the  $\mathfrak{sl}_{N-1}$  evaluation of MOY graphs.

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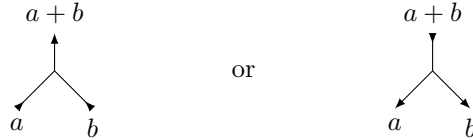
## 2. EVALUATION OF $\mathfrak{sl}_N$ -WEBS

**Definition 2.1.** Let  $N$  be a positive integer. An  $\mathfrak{sl}_N$ -web or simply *web* is an oriented, trivalent, plane graph  $\Gamma = (V, E)$  with possibly some vertex-less loops, whose edges are labeled with elements of  $\mathbb{Z}^1$ , such that for every vertex  $v$  of  $\Gamma$ , we have:

$$\sum_{v \xrightarrow{e} \bullet} \lambda(e) = \sum_{v \xleftarrow{e} \bullet} \lambda(e) \pmod{N},$$

where  $\lambda : E \rightarrow \llbracket 1, N \rrbracket$  is the labelling of the edges. Two  $\mathfrak{sl}_N$ -webs  $\Gamma_1$  and  $\Gamma_2$  are considered to be equivalent if one can obtain one from the other by reversing the orientations of some edges  $(e)_{e \in E'}$  and replacing their labels by  $N - \lambda(e)$ .

**Definition 2.2.** A *MOY graph* is a trivalent oriented plane graph  $\Gamma$  with possibly some vertex-less loops, whose edges are labeled with elements of  $\mathbb{N}$ , such that for every vertex the labels and the orientations look locally like:



If  $\Gamma$  is a MOY graph, the *writhe* of  $\Gamma$ , denoted by  $w(\Gamma)$ , is the algebraic number of circles obtained, when one replaces every edge with label  $i$  of  $\Gamma$  by  $i$  parallel copies (see Figure 1).

<sup>1</sup>Actually, if the label of an edge is not in  $\llbracket 0, N \rrbracket := \{0, 1, \dots, N\}$ , the web will be pretty uninteresting. However, for technical reasons, it is convenient to allow labelling by all relative integers.

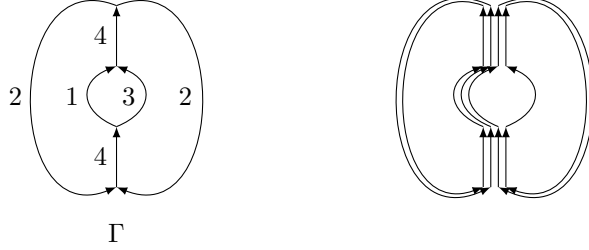


FIGURE 1. Computation of the writhe of a MOY graph: here we have  $w(\Gamma) = 2 - 2 = 0$

**Definition 2.3.** A *coloring* of a web  $\Gamma = (V, E)$  is a function  $c$  from  $E$  to  $\mathfrak{P}([1, N])$  such that:

- (E) for every edge  $e$ ,  $\#c(e) = \lambda(e)$ ,
- (V) for every vertex  $v$ , the multiset:

$$\bigcup_{v \xrightarrow{e} \bullet} c(e) \cup \bigcup_{v \xleftarrow{e} \bullet} \overline{c(e)}$$

is a multiple of  $[1, N]$ , where  $\overline{c(e)}$  is the complement of  $c(e)$  in  $[1, N]$ .

If two webs  $\Gamma_1$  and  $\Gamma_2$  are equivalent and  $c_1$  is a coloring of  $\Gamma_1$ , there is a canonical coloring of  $\Gamma_2$  obtained by replacing  $c_1(e)$  by its complement in  $[1, N]$  for every edge  $e$  whose orientation has been reversed.

**Remark 2.4.** (1) If some labels of an  $\mathfrak{sl}_N$ -web  $\Gamma$  are not in  $[0, N]$ , then the web  $\Gamma$  admit no coloring.  
 (2) If  $\Gamma$  is a MOY graph (and hence can be thought of as an  $\mathfrak{sl}_N$ -web for all  $N$ ), the condition (V) of a coloring is equivalent to saying that around each vertices, the colors of the two edges with the smallest labels form a partition of the color of the edge with the greatest label. Therefore, for each element  $i$  in  $[1, N]$ , the flow of  $i$  is preserved around each vertex and one can see the coloring of  $\Gamma$  as a collection of connected cycles colored by element of  $[1, N]$  such that:

- any two cycles with the same colors are disjoint,
- an edge  $e$  belongs to exactly  $\lambda(e)$  cycles.

**Definition 2.5.** A *bicolor*  $b$  is a subset of  $[1, N]$  with exactly two elements. The greatest color (for the natural order on  $[1, N]$ ) is denoted by  $b^+$ , the other one by  $b^-$ . If  $(\Gamma, c)$  is a colored web and  $b$  is a bicolor, the *state*  $(\Gamma, c)_b$  is the collection of oriented circles obtained from  $\Gamma$  by erasing every edge  $e$  such that the cardinal of the intersection of  $c(e)$  and  $b$  is different from 1 and reversing the orientation of every edge  $e$  such that  $\lambda(e) \cap b = \{b^-\}$ . The *degree*  $d(\Gamma, c)_b$  of a state  $(\Gamma, c)_b$  is equal to the algebraic number of circles in  $(\Gamma, c)_b$ . The degree  $d(\Gamma, c)$  of a colored web  $(\Gamma, c)$  is the sum of the degree of all the possible states.

**Theorem 2.6.** Let  $\Gamma$  be a web, the evaluation  $\langle \Gamma \rangle$  of  $\Gamma$  given in [MOY98] is equal to:

$$\langle \Gamma \rangle_{\text{col}} := \sum_{c \text{ coloring of } \Gamma} q^{d(\Gamma, c)}.$$

**Remark 2.7.** (1) It is worthwhile to note that, if  $\Gamma_1$  and  $\Gamma_2$  are two equivalent webs,  $c_1$  a coloring of  $\Gamma_1$  and  $c_2$  the corresponding coloring of  $\Gamma_2$ , then for

every bicolor  $b$ , the states  $(\Gamma_1, c_1)_b$  and  $d(\Gamma_2, c_2)_b$  are equal. It follows that  $d(\Gamma_1, c_1) = d(\Gamma_2, c_2)$  and  $\langle \Gamma_1 \rangle_{\text{col}} = \langle \Gamma_2 \rangle_{\text{col}}$ .

- (2) Let  $\Gamma$  be a web. The graph  $\Gamma'$  obtained by removing all edges<sup>2</sup> labelled by 0 or  $N$  is a web and we have  $\langle \Gamma \rangle_{\text{col}} = \langle \Gamma' \rangle_{\text{col}}$ .
- (3) The theorem can be seen as a generalization of [Rob13, Theorem 1.11], one can therefore wonder if the other result of [Rob13] remains true in our context, namely: are all colorings of a web  $\Gamma$  Kempe equivalent?

The evaluation of MOY graphs  $\langle \cdot \rangle$  is multiplicative with respect to the disjoint union and it satisfies the following skein relations:

$$(1) \quad \left\langle \begin{array}{c} \text{circle with arrow } k \end{array} \right\rangle = \binom{N}{k}_q.$$

$$(2) \quad \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \downarrow \quad \searrow \\ \quad j+k \end{array} \right\rangle = \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \downarrow \quad \searrow \\ \quad i+j \end{array} \right\rangle$$

$$(3) \quad \left\langle \begin{array}{c} m+n \\ \uparrow \quad \downarrow \\ m \quad n \\ \uparrow \quad \downarrow \\ m+n \end{array} \right\rangle = \binom{m+n}{m}_q \left\langle \begin{array}{c} \uparrow \\ m+n \end{array} \right\rangle$$

$$(4) \quad \left\langle \begin{array}{c} m \\ \uparrow \quad \downarrow \\ m+n \quad n \\ \uparrow \quad \downarrow \\ m \end{array} \right\rangle = \binom{N-m}{n}_q \left\langle \begin{array}{c} \uparrow \\ m \end{array} \right\rangle$$

$$(5) \quad \left\langle \begin{array}{c} 1 \quad m \\ \uparrow \quad \downarrow \\ m+1 \quad 1 \\ \uparrow \quad \downarrow \\ 1 \quad m \end{array} \right\rangle = \left\langle \begin{array}{c} \uparrow \\ 1 \end{array} \right\rangle \left\langle \begin{array}{c} \downarrow \\ m \end{array} \right\rangle + [N-m-1]_q \left\langle \begin{array}{c} 1 \quad m \\ \swarrow \quad \downarrow \quad \searrow \\ m-1 \quad 1 \end{array} \right\rangle$$

$$(6) \quad \left\langle \begin{array}{c} l \quad m \\ \uparrow \quad \downarrow \\ l+n \quad m-n \\ \uparrow \quad \downarrow \\ 1 \quad m+l-1 \end{array} \right\rangle = \binom{m-1}{n}_q \left\langle \begin{array}{c} l \quad m \\ \uparrow \quad \downarrow \\ l-1 \quad m+l-1 \end{array} \right\rangle + \binom{m-1}{n-1}_q \left\langle \begin{array}{c} l \quad m \\ \swarrow \quad \downarrow \quad \searrow \\ l+m \quad 1 \end{array} \right\rangle$$

$$(7) \quad \left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \downarrow \\ n+k \quad m+l-k \\ \uparrow \quad \downarrow \\ n \quad m+l \end{array} \right\rangle = \sum_{j=0}^{\infty} \binom{l}{k-j}_q \left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \downarrow \\ m-j \quad n+l+j \\ \uparrow \quad \downarrow \\ n \quad m+l \end{array} \right\rangle$$

In the previous formulas,  $q$  is a formal variable,  $[k]_q := \frac{q^{+k} - q^{-k}}{q^{+1} - q^{-1}}$  and

$$\binom{k}{l}_q := \begin{cases} 0 & \text{if } k < 0, l < 0 \text{ or } l > k, \\ \frac{[k]_q!}{[l]_q! [l-k]_q!} & \text{else,} \end{cases} \quad \text{where } [i]_q! = \prod_{j=1}^i [j]_q.$$

<sup>2</sup>Formally, one should as well remove the vertices such that all adjacent edges are labelled by 0 or  $N$ .

In the sequel, we will often write  $\binom{l}{k \ l-k}_q$  for  $\binom{k}{l}_q$  to emphasize the symmetry between  $k$  and  $l-k$  in the definition.

Wu [Wu14] proved that these relations characterize  $\langle \cdot \rangle$ . Hence to prove Theorem 2.6, it is enough to prove that  $\langle \cdot \rangle_{\text{col}}$  satisfies the same local relations.

In order to check the skein relation, it is convenient to have a local definition for  $\langle \cdot \rangle_{\text{col}}$ . For this purpose, we need some definitions:

**Definition 2.8.** An *open web* is a generic intersection of a web  $\Gamma$  with  $\mathbb{R} \times [0, 1]$  where  $I$  is a non-empty interval. By "generic" we mean that:

- $\mathbb{R} \times \{0, 1\}$  does not intersect any vertex of  $\Gamma$ .
- the intersection of  $\mathbb{R} \times I$  and the edges of  $\Gamma$  are transverse.

The *boundary*  $\partial\Gamma$  of  $\Gamma$  consists of intersection of  $\Gamma$  with  $\mathbb{R} \times \partial[0, 1]$  together with the orientations and the labels induced by  $\Gamma$ . A *coloring* of an open web is defined exactly as for webs. A coloring  $c$  of an open web induces a coloring  $c_\partial$  of its boundary.

It is worthwhile to remark that one may stack<sup>3</sup> two open webs (with some compatibility conditions of part of the boundaries) to obtain a new open web. If the colorings agree on the common boundaries, one can even stack colored webs.

**Definition 2.9.** If  $(\Gamma, c)$  is a colored open web and  $b$  is a bicolor, the *state*  $(\Gamma, c)_b$  is the collection of oriented circle and arcs obtained from  $\Gamma$  by erasing all edges  $e$  of  $\Gamma$  such that  $\#(c(e) \cap b) \neq 1$  and reversing the orientation of every remaining edge  $e$  such that  $b^- \in c(e)$ . The *degree*  $d(\Gamma, c)_b$  of the state  $(\Gamma, c)_b$  is defined by:

$$d(\Gamma, c)_b := C_+ - C_- + \frac{1}{2}(TR - TL - BR + BL)$$

where:

- $C_+$  and  $C_-$  are the numbers of positively and negatively oriented circles.
- $TR$  is the number of arcs with both ends on the top (i. e. in  $\mathbb{R} \times \{1\}$ ) rightwards oriented.
- $TL$  is the number of arcs with both ends on the top leftwards oriented.
- $BR$  is the number of arcs with both ends on the bottom (i. e. in  $\mathbb{R} \times \{0\}$ ) rightwards oriented.
- $TL$  is the number of arcs with both ends on the bottom leftwards oriented.

The *degree* of a colored web  $(\Gamma, c)$  is equal to  $d(\Gamma, c) := \sum_{b \text{ bicolor}} d(\Gamma, c)_b$ .

If  $\Gamma$  is an open web and  $c_\partial$  a coloring of its boundary, we define:

$$\langle \Gamma \rangle_{\text{col}}^{c_\partial} := \sum_{\substack{c \text{ coloring of } \Gamma \\ c \text{ induces } c_\partial \text{ on the boundary}}} q^{d(\Gamma, c)}.$$

**Lemma 2.10.** Let  $(\Gamma, c)$  be a colored open web obtained by stacking  $(\Gamma', c')$  and  $(\Gamma'', c'')$  one onto the other. For every bicolor  $b$ , we have  $d((\Gamma, c)_b) = d((\Gamma', c')_b) + d((\Gamma'', c'')_b)$ . It follows that we have:  $d(\Gamma, c) = d(\Gamma', c') + d(\Gamma'', c'')$ .

*Sketch of the proof.* If we fix a bicolor  $b$ , the degree  $d((\Gamma, c)_b)$  can be seen as the integral of a (normalized) curvature along the arcs. With this point of view, the lemma simply follows from the Chasles relation.  $\square$

**Remark 2.11.** From Lemma 2.10, we deduce that in order to check that  $\langle \cdot \rangle_{\text{col}}$  satisfies the relations (1) to (7), it is enough to check that  $\langle \cdot \rangle_{\text{col}}^{c_\partial}$  satisfies the relations (1) to (7) and for every coloring  $c_\partial$  of the common<sup>4</sup> boundary.

<sup>3</sup>Or compose, if one think of open webs as morphism in a suitable category.

<sup>4</sup>All the webs involved in a local relation have the same boundary.

3. PARTITIONS AND  $q$ -IDENTITIES

The aim of this section is to introduce degrees of partitions of ordered set and to prove Lemma 3.5 from which Theorem 2.6 will follow.

**Definition 3.1.** Let  $(X, <)$  be a finite totally ordered set and  $Y$  and  $Z$  two disjoint subsets of  $X$ . The *degree*  $d(Y \sqcup Z)$  of  $Y \sqcup Z$  is the integer defined by the formula:

$$d(Y \sqcup Z) = \#\{(y, z) \in (Y \times Z) \mid y < z\} - \#\{(y, z) \in (Y \times Z) \mid y > z\}.$$

**Lemma 3.2.** Let  $n$  and  $m$  be two integers such that  $m + n \geq 1$ . The following relation holds:

$$\binom{m+n}{m \ n}_q = q^{+m} \binom{m+n-1}{m \ n-1}_q + q^{-n} \binom{m+n-1}{m-1 \ n}_q$$

*Proof.* If  $m$  or  $n$  is negative, the relation reads  $0 = 0$ . We suppose that  $m$  and  $n$  are non-negative. This can be thought of in terms of degree of partition. The left-hand side counts with degree all the partitions of  $\llbracket 1, m+n \rrbracket$  which consists of a set  $Y$  of  $m$  elements and a set  $Z$  of  $n$  elements. The first term of the right-hand side counts partitions such that 1 is in  $Y$ , the second counts partitions such that 1 is in  $Z$ . We can as well prove this equality directly:

$$\begin{aligned} \binom{m+n}{m \ n}_q &= \frac{[m+n]!}{[m]![n]!} = \frac{(q^{+m}[n] + q^{-n}[m])[m+n-1]!}{[m]![n]!} \\ &= \frac{(q^{+m}[n])[m+n-1]!}{[m]![n]!} + \frac{(q^{-n}[m])[m+n-1]!}{[m]![n]!} \\ &= \frac{(q^{+m})[m+n-1]!}{[m]![n-1]!} + \frac{(q^{-n})[m+n-1]!}{[m-1]![n]!} \\ &= q^{+m} \binom{m+n-1}{m \ n-1}_q + q^{-n} \binom{m+n-1}{m-1 \ n}_q \end{aligned}$$

□

Note that  $\binom{m+n}{m \ n}_q$  is entirely determined by the formula of Lemma 3.2 and

the fact that for all  $k \geq 0$ ,  $\binom{k}{0 \ k}_q = \binom{k}{k \ 0}_q = 1$ .

The following lemma is not strictly necessary, however we do think that it enlightens the relation between degree of disjoint union and quantum binomials.

**Lemma 3.3.** We consider  $(X, <)$  a finite totally ordered set with  $m+n$  element. Let  $\mathcal{P}_{m,n}(X)$  the set of partition  $Y \sqcup Z$  of  $X$  such that  $\#Y = m$  and  $\#Z = n$ . The following relation holds:

$$\sum_{Y \sqcup Z \in \mathcal{P}_{m,n}(X)} q^{d(Y \sqcup Z)} = \binom{m+n}{m \ n}_q$$

*Proof.* The statement actually does not depends on  $X$ . Hence we may suppose that  $X = \llbracket 1, m+n \rrbracket$  with the natural order. Let us write:

$$p_{m,n} = \sum_{Y \sqcup Z \in \mathcal{P}_{m,n}(\llbracket 1, m+n \rrbracket)} q^{d(Y \sqcup Z)} = \binom{m+n}{m \ n}_q$$

For every positive integer  $k$ , we have  $p_{k,0} = p_{0,k} = 1$ . We have:

$$p_{m+1,n+1} = \sum_{Y \sqcup Z \in \mathcal{P}_{m+1,n+1}(\llbracket 1, m+n+2 \rrbracket)} q^{d(Y \sqcup Z)}$$

$$\begin{aligned}
&= \sum_{\substack{Y \sqcup Z \in \mathcal{P}_{m+1, n+1}([1, m+n+2]) \\ m+n+2 \in Y}} q^{d(Y \sqcup Z)} + \sum_{\substack{Y \sqcup Z \in \mathcal{P}_{m+1, n+1}([1, m+n+2]) \\ m+n+2 \in Z}} q^{d(Y \sqcup Z)} \\
&= \sum_{Y \sqcup Z \in \mathcal{P}_{m, n+1}([1, m+n+1])} q^{d(Y \sqcup Z) - (n+1)} + \sum_{\substack{Y \sqcup Z \in \mathcal{P}_{m+1, n}([1, m+n+1]) \\ m+n+1 \in Y}} q^{d(Y \sqcup Z) + m+1} \\
&= q^{n+1} p_{m, n+1} + q^{m+1} p_{m+1, n}
\end{aligned}$$

It satisfies the same recursion formula so the quantum binomial (and have the same initial values). This proves that for all  $m$  and  $n$  we have  $p_{m, n} = \binom{m+n}{m}_q$   $\square$

The following observation will be very useful for proving Lemma 3.5.

**Lemma 3.4.** *Let  $X$  and  $Y$  be two disjoint subsets of  $\llbracket 1, M \rrbracket$  and  $k$  be an integer of  $\llbracket 1, M-1 \rrbracket$ . Let us write  $X_1 = X \cap \llbracket 1, k \rrbracket$ ,  $Y_1 = Y \cap \llbracket 1, k \rrbracket$ ,  $X_2 = X \cap \llbracket k+1, M \rrbracket$  and  $Y_2 = Y \cap \llbracket k+1, M \rrbracket$ . The following relation holds:*

$$d(X \sqcup Y) = d(X_1 \sqcup Y_1) + d(X_2 \sqcup Y_2) + \#X_1 \#Y_2 - \#Y_1 \#X_2.$$

*Proof.* It follows from the definition.  $\square$

The following lemma is the key ingredient to prove theorem 2.6. It should be compared to [CKM14, Proof of relation 2.10].

**Lemma 3.5.** *Let us fix  $X$  and  $Y$  two disjoint subsets of in  $\llbracket 1, M \rrbracket$ , such that  $\#X = \#Y + l$  with  $l \geq 0$ . For every integer  $k_1$ , the following relation holds:*

$$\sum_{\substack{X = X_1 \sqcup X_2 \\ \#X_1 = k_1}} q^{d(X_1 \sqcup X_2) + d(Y \sqcup X_1)} = \sum_{j_2=0}^{\infty} \binom{l}{k_1 - j_2 \quad l - k_1 + j_2}_q \sum_{\substack{Y = Y_1 \sqcup Y_2 \\ \#Y_2 = j_2}} q^{d(Y_1 \sqcup Y_2) + d(Y_2 \sqcup X)}$$

*Proof.* The proof is done by induction on the cardinal of  $\#X + \#Y$ . If  $\#X + \#Y = 0$ , the relation reads  $1 = 1$ . For the induction, we need to be careful and stay in the case where  $\#X \geq \#Y$ . Suppose that  $\#X - \#Y \geq 1$ , then removing the smallest element of  $X \sqcup Y$  gives us two sets  $X'$  and  $Y'$  such that  $\#X' - \#Y' \geq 0$ . Let us now consider the extreme case, where  $\#X = \#Y \geq 1$ . We can distinguish two situations:

- The lowest (or the highest) element of  $X \sqcup Y$  is in  $Y$ , then removing this element gives us two sets  $X'$  and  $Y'$  such that  $\#X - \#Y = 1$ .
- The lowest element and the highest element of  $X \sqcup Y$  are in  $Y$ , in this case we can find an element  $k$  in  $\llbracket 1, M \rrbracket$  such that, if we define  $X' := X \cap \llbracket 1, k \rrbracket$ ,  $Y' := Y \cap \llbracket 1, k \rrbracket$ ,  $X'' := X \cap \llbracket k+1, M \rrbracket$ ,  $Y'' := Y \cap \llbracket k+1, M \rrbracket$ , we have  $\#X' = \#Y'$ ,  $\#X'' = \#Y''$  and none of these sets is empty.

We first suppose that  $\#X - \#Y = l \geq 1$  and the lowest element  $t$  of  $X \sqcup Y$  is in  $X$ . Let us write  $X' := X \setminus \{t\}$ . We have:

$$\begin{aligned}
&\sum_{\substack{X = X_1 \sqcup X_2 \\ \#X_1 = k_1}} q^{d(X_1 \sqcup X_2) + d(Y \sqcup X_1)} \\
&= \sum_{\substack{X = X_1 \sqcup X_2 \\ \#X_1 = k_1 \\ t \in X_1}} q^{d(X_1 \sqcup X_2) + d(Y \sqcup X_1)} + \sum_{\substack{X = X_1 \sqcup X_2 \\ \#X_1 = k_1 \\ t \in X_2}} q^{d(X_1 \sqcup X_2) + d(Y \sqcup X_1)} \\
&= \sum_{\substack{X' = X'_1 \sqcup X'_2 \\ \#X'_1 = k_1 - 1}} q^{d(X'_1 \sqcup X'_2) + d(Y \sqcup X'_1) + (\#X - k_1) - (\#X - l)} + \sum_{\substack{X' = X'_1 \sqcup X'_2 \\ \#X_1 = k_1}} q^{d(X'_1 \sqcup X'_2) + d(Y \sqcup X'_1) - k_1}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j_2=0}^{\infty} q^{l-k_1} \binom{l-1}{k_1-1-j_2 \quad l-1-(k_1-1)+j_2}_q \sum_{\substack{Y=Y_1 \sqcup Y_2 \\ \#Y_2=j_2}} q^{d(Y_1 \sqcup Y_2)+d(Y_2 \sqcup X')} \\
&\quad + \sum_{j_2=0}^{\infty} q^{-k_1} \binom{l-1}{k_1-j_2 \quad l-1-k_1+j_2}_q \sum_{\substack{Y=Y_1 \sqcup Y_2 \\ \#Y_2=j_2}} q^{d(Y_1 \sqcup Y_2)+d(Y_2 \sqcup X')} \\
&= \sum_{j_2=0}^{\infty} \left( q^{l-k_1} \binom{l-1}{k_1-1-j_2 \quad l-k_1+j_2}_q + q^{-k_1} \binom{l-1}{k_1-j_2 \quad l-1-k_1+j_2}_q \right) \\
&\quad \cdot \sum_{\substack{Y=Y_1 \sqcup Y_2 \\ \#Y_2=j_2}} q^{d(Y_1 \sqcup Y_2)+d(Y_2 \sqcup X')} \\
&= \sum_{j_2=0}^{\infty} q^{-j_2} \binom{l}{k_1-j_2 \quad l-k_1+j_2}_q \sum_{\substack{Y=Y_1 \sqcup Y_2 \\ \#Y_2=j_2}} q^{d(Y_1 \sqcup Y_2)+d(Y_2 \sqcup X')} \\
&= \sum_{j_2=0}^{\infty} \binom{l}{k_1-j_2 \quad l-k_1+j_2}_q \sum_{\substack{Y=Y_1 \sqcup Y_2 \\ \#Y_2=j_2}} q^{d(Y_1 \sqcup Y_2)+d(Y_2 \sqcup X)}
\end{aligned}$$

The case where the greatest element of  $X \sqcup Y$  is in  $X$  is analogue. We suppose now that  $\#X - \#Y = l \geq 0$  and the lowest element  $t$  of  $X \sqcup Y$  is in  $Y$ . Let us write  $Y' := Y \setminus \{t\}$ . We have:

$$\begin{aligned}
&\sum_{j_2=0}^{\infty} \binom{l}{k_1-j_2 \quad l-k_1+j_2}_q \sum_{\substack{Y=Y_1 \sqcup Y_2 \\ \#Y_2=j_2}} q^{d(Y_1 \sqcup Y_2)+d(Y_2 \sqcup X)} \\
&= \sum_{j_2=0}^{\infty} \binom{l}{k_1-j_2 \quad l-k_1+j_2}_q \sum_{\substack{Y=Y_1 \sqcup Y_2 \\ \#Y_2=j_2 \\ t \in Y_1}} q^{d(Y_1 \sqcup Y_2)+d(Y_2 \sqcup X)} \\
&\quad + \sum_{j_2=0}^{\infty} \binom{l}{k_1-j_2 \quad l-k_1+j_2}_q \sum_{\substack{Y=Y_1 \sqcup Y_2 \\ \#Y_2=j_2 \\ t \in Y_2}} q^{d(Y_1 \sqcup Y_2)+d(Y_2 \sqcup X)} \\
&= \sum_{j_2=0}^{\infty} \binom{l}{k_1-j_2 \quad l-k_1+j_2}_q \sum_{\substack{Y'=Y'_1 \sqcup Y'_2 \\ \#Y'_2=j_2}} q^{d(Y'_1 \sqcup Y'_2)+d(Y'_2 \sqcup X)+j_2} \\
&\quad + \sum_{j_2=0}^{\infty} \binom{l}{k_1-j_2 \quad l-k_1+j_2}_q \sum_{\substack{Y'=Y'_1 \sqcup Y'_2 \\ \#Y'_2=j_2-1}} q^{d(Y'_1 \sqcup Y'_2)+d(Y'_2 \sqcup X)-(\#X-l-j_2)+\#X} \\
&= \sum_{j_2=0}^{\infty} \binom{l}{k_1-j_2 \quad l-k_1+j_2}_q \sum_{\substack{Y'=Y'_1 \sqcup Y'_2 \\ \#Y'_2=j_2}} q^{d(Y'_1 \sqcup Y'_2)+d(Y'_2 \sqcup X)+j_2} \\
&\quad + \sum_{j_2=0}^{\infty} \binom{l}{k_1-(j_2+1) \quad l-k_1+(j_2+1)}_q \sum_{\substack{Y'=Y'_1 \sqcup Y'_2 \\ \#Y'_2=j_2}} q^{d(Y'_1 \sqcup Y'_2)+d(Y'_2 \sqcup X)+l+j_2+1} \\
&= \sum_{j_2=0}^{\infty} \left( q^{j_2} \binom{l}{k_1-j_2 \quad l-k_1+j_2}_q + q^{l+j_2+1} \binom{l}{k_1-(j_2+1) \quad l-k_1+(j_2+1)}_q \right)
\end{aligned}$$



$$\begin{aligned}
& \cdot \sum_{\substack{Y=Y_1 \sqcup Y_2 \\ \#Y_2'=j_2}} q^{d(Y_1 \sqcup Y_2)+d(Y_2 \sqcup X)+l+j_2+1} \\
&= \sum_{j_2=0}^{\infty} q^{k_1} \binom{l+1}{k_1-j_2 \quad l+1-k_1+j_2}_q \sum_{\substack{Y=Y_1 \sqcup Y_2 \\ \#Y_2'=j_2}} q^{d(Y_1 \sqcup Y_2)+d(Y_2 \sqcup X)+l+j_2+1} \\
&= \sum_{\substack{X=X_1 \sqcup X_2 \\ \#X_1=k_1}} q^{d(X_1 \sqcup X_2)+d(Y' \sqcup X_1)+k_1} \\
&= \sum_{\substack{X=X_1 \sqcup X_2 \\ \#X_1=k_1}} q^{d(X_1 \sqcup X_2)+d(Y \sqcup X_1)}
\end{aligned}$$

The case were the greatest element of  $X \sqcup Y$  is in  $Y$  is analogue.

We suppose now that the greatest and the lowest element of  $X \sqcup Y$  are in  $X$  and that  $l = 0$ . We use the notations explained before and we write  $k' = \#X' = \#Y'$  and  $k'' = \#X'' = \#Y''$ . For readability it is convenient to set  $l' = l'' = 0$ . We have:

$$\begin{aligned}
& \sum_{j_2=0}^{\infty} \binom{l}{k_1-j_2 \quad l-k_1+j_2}_q \sum_{\substack{Y=Y_1 \sqcup Y_2 \\ \#Y_2=j_2}} q^{d(Y_1 \sqcup Y_2)+d(Y_2 \sqcup X)} = \sum_{\substack{Y=Y_1 \sqcup Y_2 \\ \#Y_2=k_1}} q^{d(Y_1 \sqcup Y_2)+d(Y_2 \sqcup X)} \\
&= \sum_{k_1'+k_1''=k_1} \sum_{\substack{Y'=Y_1' \sqcup Y_2' \\ \#Y_2'=k_1' \\ Y''=Y_1'' \sqcup Y_2'' \\ \#Y_2''=k_1''}} q^{d(Y_1' \sqcup Y_2')+d(Y_2' \sqcup X')+d(Y_1'' \sqcup Y_2'')+d(Y_2'' \sqcup X'')+(k'-k_1')k_1''-k_1'(k''-k_1'')+k_1'k''-k_1'k_1''} \\
&= \sum_{k_1'+k_1''=k_1} \sum_{\substack{Y'=Y_1' \sqcup Y_2' \\ \#Y_2'=k_1' \\ Y''=Y_1'' \sqcup Y_2'' \\ \#Y_2''=k_1''}} q^{d(Y_1' \sqcup Y_2')+d(Y_2' \sqcup X')+d(Y_1'' \sqcup Y_2'')+d(Y_2'' \sqcup X'')} \\
&= \sum_{k_1'+k_1''=k_1} \left( \sum_{\substack{Y'=Y_1' \sqcup Y_2' \\ \#Y_2'=k_1'}} q^{d(Y_1' \sqcup Y_2')+d(Y_2' \sqcup X')} \right) \left( \sum_{\substack{Y''=Y_1'' \sqcup Y_2'' \\ \#Y_2''=k_1''}} q^{d(Y_1'' \sqcup Y_2'')+d(Y_2'' \sqcup X'')} \right) \\
&= \sum_{k_1'+k_1''=k_1} \left( \sum_{j_2'=0}^{\infty} \binom{l'}{k_1'-j_2' \quad l'-k_1'+j_2'}_q \sum_{\substack{Y'=Y_1' \sqcup Y_2' \\ \#Y_2'=j_2'}} q^{d(Y_1' \sqcup Y_2')+d(Y_2' \sqcup X')} \right) \\
& \quad \left( \sum_{j_2''=0}^{\infty} \binom{l''}{k_1''-j_2'' \quad l''-k_1''+j_2''}_q \sum_{\substack{Y''=Y_1'' \sqcup Y_2'' \\ \#Y_2''=j_2''}} q^{d(Y_1'' \sqcup Y_2'')+d(Y_2'' \sqcup X'')} \right) \\
&= \sum_{k_1'+k_1''=k_1} \left( \sum_{\substack{X=X_1' \sqcup X_2' \\ \#X_1'=k_1'}} q^{d(X_1' \sqcup X_2')+d(Y' \sqcup X_1')} \right) \left( \sum_{\substack{X''=X_1'' \sqcup X_2'' \\ \#X_1''=k_1''}} q^{d(X_1'' \sqcup X_2'')+d(Y'' \sqcup X_1'')} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k'_1 + k''_1 = k_1} \sum_{\substack{X = X'_1 \sqcup X'_2 \\ \#X'_1 = k'_1 \\ X'' = X'_1 \sqcup X'_2 \\ \#X'_1 = k''_1}} q^{d(X'_1 \sqcup X'_2) + d(Y' \sqcup X'_1) + d(X''_1 \sqcup X''_2) + d(Y'' \sqcup X''_1)} \\
&= \sum_{k'_1 + k''_1 = k_1} \sum_{\substack{X = X'_1 \sqcup X'_2 \\ \#X'_1 = k'_1 \\ X'' = X'_1 \sqcup X'_2 \\ \#X'_1 = k''_1}} q^{d(X'_1 \sqcup X'_2) + d(Y' \sqcup X'_1) + d(X''_1 \sqcup X''_2) + d(Y'' \sqcup X''_1) + k'_1(k'' - k'_1) - (k' - k'_1)k''_1 + k'k''_1 - k'_1k''} \\
&= \sum_{\substack{X = X_1 \sqcup X_2 \\ \#X_1 = k_1}} q^{d(X_1 \sqcup X_2) + d(Y \sqcup X_1)}
\end{aligned}$$

□

## 4. CHECKING THE SKEIN RELATIONS

**Lemma 4.1.** *Let  $\Gamma$  be a web, and  $\Gamma'$  the web obtained by deleting all the edges labelled by 0 or  $N$ . We have:*

$$\langle \Gamma \rangle_{\text{col}} = \langle \Gamma' \rangle_{\text{col}}.$$

*Proof.* We clearly have a one-one correspondence between the colorings of  $\Gamma$  and of  $\Gamma'$  since an edge labeled 0 can only be colored by  $\emptyset$  and an edge labeled  $N$  can only be colored by  $\llbracket 1, N \rrbracket$ . Furthermore, the edges of  $\Gamma$  labeled by 0 and  $N$  never appear in any state for any coloring, this means that the state  $(\Gamma, c)_b$  and  $(\Gamma', c')_b$  are equal (where  $c$  and  $c'$  are two colorings corresponding one to another). This proves that  $\langle \Gamma \rangle_{\text{col}} = \langle \Gamma' \rangle_{\text{col}}$ . □

**Lemma 4.2.** *It is enough to check relations (2) and (7).*

*Proof.* We prove that (1) and (4) follow from (3). We suppose that relation (3) holds. We have

$$\left\langle \begin{array}{c} \text{circle with arrow} \\ k \end{array} \right\rangle_{\text{col}} = \left\langle \begin{array}{c} \text{circle with arrows} \\ N-k \quad k \\ N \end{array} \right\rangle_{\text{col}} = \begin{pmatrix} N & \\ k & N-k \end{pmatrix}_q.$$

This proves relation (1) holds.

$$\begin{aligned}
\left\langle \begin{array}{c} \text{circle with arrows} \\ m+n \quad n \\ m \end{array} \right\rangle_{\text{col}} &= \left\langle \begin{array}{c} \text{circle with arrows} \\ N-m \quad n \\ N-m \end{array} \right\rangle_{\text{col}} = \begin{pmatrix} N-m & \\ n & N-m-n \end{pmatrix}_q \left\langle \begin{array}{c} \text{vertical arrow} \\ N-m \end{array} \right\rangle_{\text{col}} \\
&= \begin{pmatrix} N-m & \\ n & N-m-n \end{pmatrix}_q \left\langle \begin{array}{c} \text{vertical arrow} \\ m \end{array} \right\rangle_{\text{col}}
\end{aligned}$$

This proves relation (4) holds.

We prove that relations (3), (5) and (6) follow from (7). We suppose that (7) holds.

Relation (6) is a special case of relation (7): by setting  $n = 1$ ,  $m = l'$ ,  $k = l' + n' - 1$  and  $l = m' - 1$  in (7), we obtain (6) with all the labels replaced by labels with  $'$ .

Relation (5) is a special case of relation (5): by setting  $l = 1$ ,  $m = N - m'$ , and  $n = N - m' - 1$  in (6), we obtain (5) with all the labels replaced by labels with  $'$ .

Relation (3) is a special case of relation (7): by setting  $m = n = 0$ ,  $l = m' + n'$  and  $k = m'$  in (7), we obtain (3) with all the labels replaced by labels with '.  $\square$

**Lemma 4.3.** *The following relation holds:*

$$\left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \quad \nearrow \\ \quad \quad j+k \\ \uparrow \\ i+j+k \end{array} \right\rangle_{\text{col}}^{c_\partial} = \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \quad \nearrow \\ i+j \quad \quad \quad \\ \uparrow \\ i+j+k \end{array} \right\rangle_{\text{col}}^{c_\partial}$$

where  $c_\partial$  is any coloring of the boundary (see footnote on page 5).

*Proof.* Let  $\Gamma$  be the open web on the left and  $\Gamma'$  the open web on the right. The boundary consists of four points,  $\tau_1, \tau_2$  and  $\tau_3$  on the top and  $\beta$  on the bottom. Let us consider a coloring  $c_\beta$  of the boundary (that is any application  $\{\tau_1, \tau_2, \tau_3, \beta\} \rightarrow \llbracket 1, N \rrbracket$  such that  $\#c_\partial(\tau_1) = i$ ,  $\#c_\partial(\tau_2) = j$ ,  $\#c_\partial(\tau_3) = k$  and  $\#c_\partial(\beta) = i + j + k$ ).

Suppose  $X_1 := c_\partial(\tau_1)$ ,  $X_2 := c_\partial(\tau_2)$  and  $X_3 := c_\partial(\tau_3)$  form a partition of  $c_\partial(\beta)$ . Then there exists a unique coloring  $c$  of  $\Gamma$  and a unique coloring  $c'$  of  $\Gamma'$  compatible with  $c_\partial$ . We need to compare the degree  $d(\Gamma, c)$  and  $d(\Gamma', c')$ . The values of  $d(\Gamma, c)_b$  and  $d(\Gamma', c')_b$  for every bicolor  $b$  are given in Table 1. We have  $d(\Gamma, c)_b = d(\Gamma', c')_b$  for all  $b$ . Hence  $d(\Gamma, c) = d(\Gamma', c')$  and  $\langle \Gamma \rangle_{\text{col}}^{c_\partial} = \langle \Gamma' \rangle_{\text{col}}^{c_\partial}$ .

Suppose  $c_\partial(\tau_1)$ ,  $c_\partial(\tau_2)$  and  $c_\partial(\tau_3)$  do not form a partition of  $c_\partial(\beta)$ . Then there is no coloring of  $\Gamma$  inducing  $c_\partial$  and no coloring of  $\Gamma'$  inducing  $c_\partial$ . Hence we have  $\langle \Gamma \rangle_{\text{col}}^{c_\partial} = \langle \Gamma' \rangle_{\text{col}}^{c_\partial} = 0$ .

Finally we have

$$\left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \quad \nearrow \\ \quad \quad j+k \\ \uparrow \\ i+j+k \end{array} \right\rangle_{\text{col}}^{c_\partial} = \left\langle \begin{array}{c} i \quad j \quad k \\ \swarrow \quad \nearrow \quad \nearrow \\ i+j \quad \quad \quad \\ \uparrow \\ i+j+k \end{array} \right\rangle_{\text{col}}^{c_\partial}$$

for all colorings  $c_\partial$  of the boundary.  $\square$

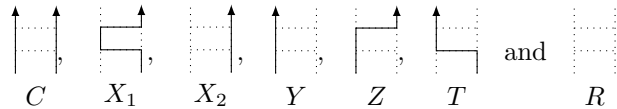
**Lemma 4.4.** *The following relation holds:*

$$\left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ n+k \quad \quad m+l-k \\ \leftarrow \quad \rightarrow \\ \quad \quad k \\ \uparrow \\ n \quad m+l \end{array} \right\rangle_{\text{col}}^{c_\partial} = \sum_{j=0}^{\infty} \binom{l}{k-j \quad l-k+j}_q \left\langle \begin{array}{c} m \quad n+l \\ \uparrow \quad \uparrow \\ m-j \quad \quad n+l+j \\ \leftarrow \quad \rightarrow \\ \quad \quad n+j-m \\ \uparrow \\ n \quad m+l \end{array} \right\rangle_{\text{col}}^{c_\partial}$$

where  $c_\partial$  is any coloring of the boundary.

*Proof.* Let us denote by  $\Gamma$  the open web on the left-hand side of the relation and by  $(\Gamma_j)_{j \in \mathbb{N}}$  the open webs on the right-hand side of the relation.

Let  $c$  be a coloring of  $\Gamma$  and  $i$  an element of  $\llbracket 1, N \rrbracket$ . We consider the set  $E_i$  of edges  $e$  of  $\Gamma$  such that  $i$  is in  $c(e)$ . Due to the flow condition on MOY graphs (see Remark 2.4), the following configurations  $E_i$  are the only possible ones (the solid edges are in  $E_i$  the others not):



This gives us a partition of  $\llbracket 1, N \rrbracket$  into 7 sets :  $C$ ,  $X_1$ ,  $X_2$ ,  $Y$ ,  $Z$ ,  $T$  and  $R$ . The coloring of the boundary induced by  $c$  is:





































$b_- \backslash b_+$	$X_1$	$X_2$	$X_3$	$R$	$b_- \backslash b_+$	$X_1$	$X_2$	$X_3$	$R$
$X_1$									
$X_2$									
$X_3$									
$R$									

TABLE 1. Computations of  $d(\Gamma, c)_b$  (on the left) and  $d(\Gamma', c')_b$  (on the right). How to read these tables: The diagram contained in the blue cell, is the state  $(\Gamma, c)_b$  when  $b_+$  is in  $X_1$  and  $b_-$  is in  $X_3$ . The number  $+\frac{1}{2}$  contained in the blue cell is  $d(\Gamma, c)_b$ . How to check these tables: There is an anti-symmetry property: if one transposed the table, one should obtain the same diagram with the orientation reversed and the degree should be multiplied by  $-1$ . There is a consistency property for each line and each column: If two diagrams are on the same line or on the same column and have a common solid edge, the orientations of this edge on the two diagrams should be the same.

- $C \sqcup Y \sqcup Z$  for the top right point,
- $C \sqcup X_1 \sqcup X_2 \sqcup T$  for the top left point,
- $C \sqcup Y \sqcup Z$  for the bottom right point,
- $C \sqcup X_1 \sqcup X_2 \sqcup Z$  for the bottom left point.

We have the following conditions on the cardinals of the sets  $C$ ,  $X_1$ ,  $X_2$ ,  $Y$ ,  $Z$ ,  $T$  and  $R$  (we name them  $c$ ,  $x_1$ ,  $x_2$ ,  $y$ ,  $z$ ,  $t$  and  $r$ ):

- $c + y + z = m$ ,
- $c + y + t = n$ ,
- $c + x_1 + x_2 + t = n + l$ ,
- $c + x_1 + x_2 + z = m + l$ ,
- $x_1 + t = k$ ,
- $c + r + x_1 + x_2 + y + z + t + r = N$ .

From this we easily deduce that we have:

$$x_1 + x_2 - y = l, \quad x_1 + x_2 = n + l - c - t \quad \text{and} \quad x_1 = k - t.$$

If we are given a coloring of the boundary (which extends to a coloring of  $\Gamma$ ), we can recover<sup>5</sup>  $C$ ,  $R$ ,  $Y$ ,  $Z$ ,  $T$  and  $X := X_1 \sqcup X_2$ .

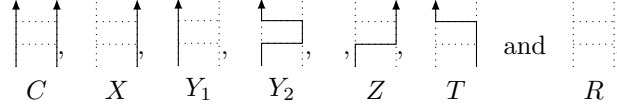
On the other hand, if we are given a partition of  $\llbracket 1, N \rrbracket$  into 6 sets  $C$ ,  $X$ ,  $Y$ ,  $Z$ ,  $T$  and  $R$  such that their cardinals  $c$ ,  $x$ ,  $y$ ,  $z$ ,  $t$  and  $r$  satisfy:

- $c + y + z = m$ ,
- $c + y + t = n$ ,
- $c + x + t = n + l$ ,
- $c + x + z = m + l$
- and  $t \leq k$ ,

<sup>5</sup>For example the set  $C$  is the intersection of the colorings of the four points and the set  $X$  is the intersection of the coloring of the two points on the right minus the set  $C$ .

every partition of  $X$  into two sets  $X_1$  and  $X_2$ , such that  $\#X_1 = k - t$  provides a coloring of  $\Gamma$ . This means, that there exist exactly  $\binom{n+l-c-t}{k-t}$  such colorings.

We now consider a coloring  $c_j$  of the open web  $\Gamma_j$ . As we did for  $\Gamma$ , we can form a partition of  $\llbracket 1, N \rrbracket$ :



The coloring of the boundary induced by  $c_j$  is:

- $C \sqcup Y_1 \sqcup Y_2 \sqcup Z$  for the top right point,
- $C \sqcup X \sqcup T$  for the top left point,
- $C \sqcup Y_1 \sqcup Y_2 \sqcup Z$  for the bottom right point,
- $C \sqcup X \sqcup Z$  for the bottom left point.

We have the following conditions on the cardinals of the sets  $C, X, Y_1, Y_2, Z, T$  and  $R$  (we name them  $c, x, y_1, y_2, z, t$  and  $r$ ):

- $c + y_1 + y_2 + z = m$ ,
- $c + y_1 + y_2 + t = n$ ,
- $c + x + t = n + l$ ,
- $c + x + z = m + l$ ,
- $y_2 + t = j$ ,
- $c + r + x_1 + x_2 + y + z + t + r = N$ .

From this we easily deduce that we have:

$$x - (y_1 + y_2) = l, \quad x = n + l - c - t \quad \text{and} \quad y_1 = j - t.$$

If we are given a coloring of the boundary (which extend to a coloring of  $\Gamma_j$ ), we can recover  $C, X, Z, T, R$  and  $Y := Y_1 \sqcup Y_2$  with the same strategy as for  $\Gamma$ .

If we are given a partition of  $\llbracket 1, N \rrbracket$  into six sets  $C, X, Y, Z, T$  and  $R$  such that their cardinals  $c, x, y, z, t$  and  $r$  satisfy

- $c + y_1 + y_2 + z = m$ ,
- $c + y_1 + y_2 + t = n$ ,
- $c + x + t = n + l$ ,
- $c + x + z = m + l$ ,
- $t \leq j$ ,

every partition of  $Y$  into two sets  $Y_1$  and  $Y_2$ , such that  $\#Y_2 = j - t$  provides a coloring of  $\Gamma_j$ . This means, that there exist exactly  $\binom{n-c-t}{j-t}$  such colorings.

Note that if  $j > k$ , the coefficient multiplying  $\Gamma_j$  is equal to zero. Hence we may suppose that  $j \leq k$ . In this case the condition for a coloring to be extendable to  $\Gamma$  is stronger than the condition to be extendable to  $\Gamma_j$ . If a coloring  $c_\partial$  does not extend to  $\Gamma$ , the equality simply says  $0 = 0$ .

Let us suppose that  $c_\partial$  extends to  $\Gamma$ . We denote  $C, R, X, Z, T, Y$  the partition of  $\llbracket 1, N \rrbracket$  such that:

- The top right point is colored by  $C \sqcup Y \sqcup Z$ ,
- The top left point is colored by  $C \sqcup X \sqcup T$ ,
- The bottom right point is colored by  $C \sqcup Y \sqcup T$ ,
- The bottom left point is colored by  $C \sqcup X \sqcup Z$ .

We have  $\#X = \#Y + l$ .

The colorings of  $\Gamma$  which induce  $c_\partial$  on the boundary are given by partitions  $X_1 \sqcup X_2$  of  $X$  such that  $X_1$  has  $k - t$  elements. If we fix  $c$  such a coloring (notations are given in Figure 2), we can compute  $d((\Gamma, c)_b)$  for every bicolor  $b$ . The computations

are done in Table 2. From the table we deduce that  $d((\Gamma, c)_b) = \delta(c_\partial) + d(X_1 \sqcup X_2) + d(Y \sqcup X_1)$  where  $\delta(c_\partial)$  is a constant depending only on  $c_\partial$ . We obtain:

$$\langle \Gamma \rangle_{\text{col}}^{c_\partial} = q^{\delta(c_\partial)} \sum_{X=X_1 \sqcup X_2 \# X_1=k-t} q^{d(X_1 \sqcup X_2) + d(Y \sqcup X_1)}.$$

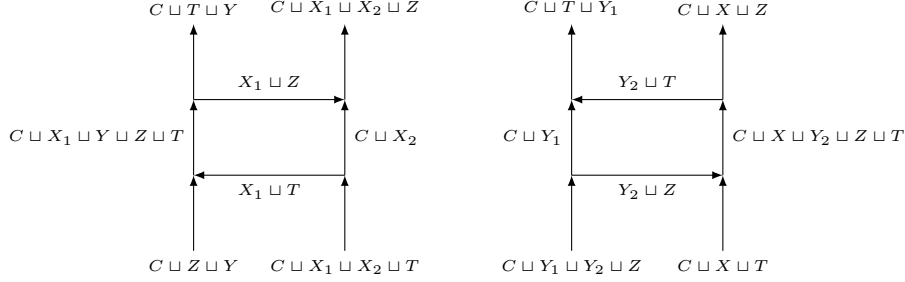


FIGURE 2. Notations for the coloring  $c$  of  $\Gamma$  and the coloring  $c_j$  of  $\Gamma_j$ .


















































$b^+ \setminus b^-$	$C$	$X_1$	$X_2$	$Y$	$Z$	$T$	$R$
$C$	 0	 0	 0	 0	 0	 0	 0
$X_1$	 0	 0	 -1	 +1	 $+\frac{1}{2}$	 $+\frac{1}{2}$	 0
$X_2$	 0	 +1	 0	 0	 $+\frac{1}{2}$	 $+\frac{1}{2}$	 0
$Y$	 0	 -1	 0	 0	 $-\frac{1}{2}$	 $-\frac{1}{2}$	 0
$Z$	 0	 $-\frac{1}{2}$	 $-\frac{1}{2}$	 $+\frac{1}{2}$	 0	 0	 0
$T$	 0	 $-\frac{1}{2}$	 $-\frac{1}{2}$	 $+\frac{1}{2}$	 0	 0	 0
$R$	 0	 0	 0	 0	 0	 0	 0

TABLE 2. Computations of  $d(\Gamma, c)_b$ . The red cells emphasize the contributions which do not only depend on  $c_\partial$ .

The colorings of  $\Gamma_j$  which induce  $c_\partial$  on the boundary are given by partitions  $Y_1 \sqcup Y_2$  of  $Y$  such that  $Y_2$  has  $j - t$  elements. If we fix  $c_j$  such a coloring (notations are given in Figure 2), we can compute  $d((\Gamma_j, c_j)_b)$  for every bicolor  $b$ . The computations are done in Table 3. From the table we deduce that  $d((\Gamma_j, c_j)_b) = \delta(c_\partial) + d(Y_1 \sqcup Y_2) + d(Y_2 \sqcup X)$  where  $\delta(c_\partial)$  is the same constant as for  $d((\Gamma, c)_b)$ . We obtain

$$\langle \Gamma_j \rangle_{\text{col}}^{c_\partial} = q^{\delta(c_\partial)} \sum_{Y=Y_1 \sqcup Y_2 \# Y_1=j-t} q^{d(Y_1 \sqcup Y_2) + d(Y_2 \sqcup X)}.$$


















































$b^+ \backslash b^-$	$C$	$X$	$Y_1$	$Y_2$	$Z$	$T$	$R$
$C$	 0	 0	 0	 0	 0	 0	 0
$X$	 0	 0	 0	 +1	 $+\frac{1}{2}$	 $+\frac{1}{2}$	 0
$Y_1$	 0	 0	 0	 -1	 $-\frac{1}{2}$	 $-\frac{1}{2}$	 0
$Y_2$	 0	 -1	 +1	 0	 $-\frac{1}{2}$	 $-\frac{1}{2}$	 0
$Z$	 0	 $-\frac{1}{2}$	 $+\frac{1}{2}$	 $+\frac{1}{2}$	 0	 0	 0
$T$	 0	 $-\frac{1}{2}$	 $+\frac{1}{2}$	 $+\frac{1}{2}$	 0	 0	 0
$R$	 0	 0	 0	 0	 0	 0	 0

TABLE 3. Computations of  $d(\Gamma_j, c_j)_b$ . The red cells emphasize the contributions which do not only depend on  $c_\partial$ .

We conclude the proof by applying Lemma 3.5 with  $k_1 = k - t$  and  $j_2 = j - t$ .  $\square$

## 5. A NEW OF SKEIN RELATION

Now that we know that  $\langle \cdot \rangle_{\text{col}}$  and  $\langle \cdot \rangle$  coincide on MOY graph, we might denote both by  $\langle \cdot \rangle_N$ . We would like now to relate  $\mathfrak{sl}_N$ -evaluations of MOY graphs for different  $N$ 's.

**Definition 5.1.** Let  $\Gamma$  be a closed MOY graph and  $A = \{\alpha_1, \dots, \alpha_k\}$  a collection of disjoint oriented cycles in  $\Gamma$ . We denote by  $\Gamma^{A_N}$  the MOY graph obtained by reversing the orientations of every edge included in  $A$  and replacing the label  $i$  of such an edge by  $N - i$ .

**Remark 5.2.** The MOY graphs  $\Gamma$  and  $\Gamma^{A_N}$  are equivalent as  $\mathfrak{sl}_N$ -webs but not as  $\mathfrak{sl}_{N-1}$ -webs (see Remark 2.7).

**Proposition 5.3.** Let  $\Gamma$  be a MOY graph. The following equality holds:

$$\begin{aligned}
 \langle \Gamma \rangle_N &= \sum_{A \text{ collection of disjoint cycles}} q^{-w(\Gamma^{A_N})} \langle \Gamma^{A_N} \rangle_{N-1} \\
 &= \sum_{A \text{ collection of cycles}} q^{+w(\Gamma^A)} \langle \Gamma^{A_N} \rangle_{N-1}.
 \end{aligned}$$

*Proof.* We only prove

$$\langle \Gamma \rangle_N = \sum_{A \text{ collection of cycles}} q^{-w(\Gamma^{A_N})} \langle \Gamma_{N-1}^{A_N} \rangle$$

since the other equality follows from the symmetry of  $\langle \cdot \rangle$  in  $q$  and  $q^{-1}$ . This equality is a consequence from the following observation: If  $c$  is a coloring of  $\Gamma$ , we can

consider the set  $A(c)$  of disjoint cycles which consists of the edges whose colorings contain  $N$ . The web  $\Gamma^{A(c)N}$  inherits a coloring  $c'$  such that none of the colors of the edges contain  $N$ . The degree  $d_N$  of the coloring  $c'$  as a coloring of an  $\mathfrak{sl}_N$ -web is equal to  $d_{N-1} - w(\Gamma^{A_N})$  where  $d_{N-1}$  is the degree of the coloring  $c'$  as a coloring of an  $\mathfrak{sl}_{N-1}$ -web:

$$d_N - d_{N-1} = \sum_{j=1}^{N-1} d(\Gamma^{A_N}, c')_{(j,N)} = -w(\Gamma^{A_N}).$$

Conversely, if  $A$  is a collection of disjoint cycles of  $\Gamma$ , and  $c'$  a coloring of  $\Gamma^{A_N}$  as a  $\mathfrak{sl}_{N-1}$ -web, the  $\mathfrak{sl}_N$ -web  $\Gamma$  inherits a coloring  $c$  and the previous equality holds.  $\square$

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